Strategic beliefs

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Abstract

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1 Introduction

In the classical financial economics theory, decision makers are assumed to have homogenous and rational expectations. This assumption has been the basis for many developments in finance. Among these developments, the portfolio selection model (Markowitz, 1952) and the Capital Asset Pricing Model (CAPM, Sharpe 1964 and Lintner 1965) play an important role. Given their simplicity and empirical tractability, these models and their subsequent extensions have become a significant cornerstone of theoretical and applied economics from insurance and financial theory to the theory of the firm.

However, the last 30 years have seen an accumulation of empirical tests that invalidate the theoretical conclusions of these models based on rational expectations. The homogeneous prior beliefs assumption (Harsanyi doctrine) is weaker than the assumption of rational expectations that all agents' prior beliefs are equal to the objective probabilities. But like rational expectations, the common priors assumption is quite restrictive and does not allow agents to "agree to disagree" (Aumann, 1976). It suffices to observe the heterogeneity of analysts or professional forecasters forecasts or more generally of experts opinions to realize that this assumption is not realistic. Savage (1954) provides axiomatic foundations for a more general theory in which agents hold arbitrary prior beliefs, so agents can agree to disagree. But the alternative to rational expectations lacks discipline and if beliefs can be arbitrary, theory provides little structure or predictive power. Indeed, once relaxed the assumption of rational and homogenous expectations, arise the following questions: How do agents form their beliefs? Do beliefs exhibit optimism? pessimism? overconfidence? doubt? How are these possible biases related to the agents preferences? How are agents' beliefs affected by strategic interaction? What is the impact of these beliefs or expectations on individual decisions? What is the impact of these beliefs on equilibrium characteristics such as prices, risk premium and risk-sharing?

We provide a discipline for beliefs formation through a model of strategic beliefs. More precisely, we consider beliefs as a strategic variable that agents can manipulate to maximize their utility from trade. Our aim is to provide a rationale for beliefs subjectivity and heterogeneity that enlightens the reflexion about the questions above.

Several models in economics study models in which forward-looking agents optimally distort beliefs (Akerlof-Dickens, 1982, Brunnermeier-Parker, 2005, Gollier-Muerman, 2006). In all these models, the beliefs result from an individual optimization problem while our model of beliefs formation is strategic. In our model, we do not impose a specific mental process linked to anticipatory feelings like Methode Coué, positive or wishful thinking, fear of disappointment, and leading systematically to a given behavioral bias.

More precisely, we suppose that agents choose strategic beliefs, so that the associated demand is optimal in the sense of a Nash equilibrium in demands, as presented in Kyle (1989). Each player can choose a belief to maximize his utility from trade, taking into account the effect his choice has on price and taking as given the strategy of the other players. We will consider two situations that lead in fact to the same conclusions. In the first situation, the set of players corresponds to the set of all investors. The adopted imperfect competition approach is then particularly significant when there is a small number of investors. This is the case, for instance, in reinsurance markets or private equity markets. This is also the case in stock markets when the capital is highly concentrated or in markets for financial products that require a high level of technicality. We can also argue that "the real issue is not so much how many investors there are, but to what extent investors cluster in their beliefs" [Shefrin, 2005, p216]. This is why we also consider a second situation where the players are "gurus" (influential investors, newsletters writers,...). Like in Benabou-Laroque (2001), the "guru" we have in mind "issues forecats but is also in the business of trading, for his own account or some investment firm." The following example quoted by Benabou-Laroque (2001) "provides the most dramatic illustration of prices reacting to someone's announcement"

> In the nervous market of 1987, Mr Prechter has emerged as both prophet and deity, an adviser whose advice reaches so many investors that he tends to pull the market the way he has predicted it will move [International Herald Tribune, October 3, 1987]

In our model, each investor adopts the announced belief of one guru. This corresponds to the findings of Fisher and Statman where it appears that "there is a positive relationship between changes in the sentiment of individual investors and that of newsletter writers". The gurus choose their "beliefs" strategically taking into account the impact of their announced beliefs on other agents hence on prices. For credibility reasons they act according to their announced beliefs. In both situations, the strategic beliefs are not true beliefs in the sense that they don't correspond to what the players truly believe but to how they behave. We consider them as beliefs because they might be interpreted as such by econometricians who observe portfolio choices. They are also interpreted as such in the model with gurus by the investors that adopt gurus' beliefs.

Our findings are the following. First, a strategic behavior leads to beliefs subjectivity and heterogeneity. This means that in a standard portfolio/equilibrium problem in which agents are allowed to manipulate their beliefs, the objective belief is not optimal, and agents differ in their optimal beliefs. Indeed, optimism (resp. overconfidence) as well as pessimism (resp. doubt) both emerge as optimal beliefs. Furthermore, we find a positive correlation between pessimism (resp. doubt) and risk tolerance. The intuition is as follows. For a very risk tolerant agent, his demand in the risky asset is positive, so that his expected utility from trade is decreasing in the price of the risky asset. The choice of a pessimistic belief is associated to a lower demand, hence to a lower price, and the optimal belief balances this benefit of pessimism against the costs of worse decision making. The converse reasoning applies to a very risk averse agent, who, at the equilibrium, has a negative demand in the risky asset and benefits from optimism. As a consequence of this positive correlation between pessimism and risk tolerance, there is less risk sharing and the volume of trade is decreased compared to the standard setting. The same applies for doubt.

Second, the representative agent belief, or the consensus belief, which is given by the average of the individual beliefs weighted by the risk tolerance, is pessimistic. Intuitively, the more risk tolerant agents make the market, and the consensus belief reflects the characteristics of the more risk tolerant. Since we have just seen that the more risk tolerant are pessimistic, it is consistent to obtain a pessimistic consensus belief. Moreover, the average (unweighted) belief is also pessimistic, which means that the pessimistic risk tolerant agents are more pessimistic than the optimistic and there is then a pessimistic bias in individual beliefs. Such a pessimistic bias is also obtained in empirical studies in a purely behavioral setting (Ben Mansour et al., 2006), in a decision theory framework (Wakker, 2001) or in a market framework (Giordani-Söderlind, 2006). In particular, as underlined by Shefrin (2005) based on Wall $treet Week data "between 1983 and 2002, professional investors were unduly pessimistic, underestimating market returns".

As a consequence of the pessimistic bias at the aggregate level, the risk premium is greater than in the standard rational expectations equilibrium. The fact that a pessimistic bias and a positive correlation between risk tolerance and pessimism lead to an increase of the market price of risk has been underlined by Abel (1989), Calvet et al. (2002), Detemple-Murthy (1994), Gollier (2007) and Jouini-Napp (2006); in their models, beliefs are exogenously given.

This increase of the market price of risk is interesting in light of the risk-premium puzzle on financial markets. In the insurance industry, our results lead to a situation where the more risk averse agent (the insured) is optimistic and the less risk averse agent (the insurer) is pessimistic. The average belief is pessimistic leading to a higher insurance premium, which might help to explain the purchase of vastly overpriced insurance in a range of situations (Cutler-Zeckhauser, 2004). In corporate finance, IPO's can be modeled as a decision for a risk averse entrepreneur to sell shares of his firm to more risk tolerant investors. The application of our results to such a setting leads to a pessimistic consensus belief. As a result, the firm is underpriced and the short run return is large, which is consistent with the empirical literature on IPO's (Ibbotson and Ritter, 1995). Obviously, we don't pretend that strategic interaction, such as in our simple model, is the unique explanation for these puzzles, however, it is interesting to remark that our model helps to explain these puzzles as well as beliefs heterogeneity without introducing any information asymmetry.

Our results obtained in a strategic interaction framework differ from those obtained in an *optimal* beliefs/illusions setting, in which there is no beliefs heterogeneity and an optimistic bias[1] (Brunnermeier-Parker, 2005, Gollier, 2005). This bias results from the specific mental process they consider. Indeed, in these models, subjective beliefs maximize the agents expected well-being defined as the time average of expected felicity over all periods. Since agents that care about future utility flows have a higher current felicity if they are optimistic, the optimal beliefs balance this benefit of optimism against the costs of worse decision making.

The paper is organized as follows. Section 2 introduces and discusses the mechanism of strategic beliefs in a simple setting. Secion 3 generalizes the model to a market with N players and considers both the case of a small number of investors and the case of a large number of investors but a small number of influential investors/gurus. Section 4 provides extensions and generalizations of the model in order to take into account more general utility functions and payoffs distributions. It also extends the results to a framework with more than one asset. Section 5 compares strategic beliefs characteristics with those of *optimal* beliefs (Brunnermeier-Parker, 2005, Gollier-Muerman, 2006). All proofs are provided in the Appendix.

[1] Gollier-Muerman (2006) consider a model of optimal beliefs with ex ante savoring and ex post disappointment. Depending upon the intensity of anticipatory feelings and disappointment they might also obtain a systematic pessimsistic bias.

2 The mechanism of strategic beliefs

In this section, we describe in a simple setting the mechanism of strategic beliefs.

We consider a standard equilibrium model, except that we allow strategic interaction to be incorporated into the analysis. More precisely, we assume that the economy is composed of two agents, who differ in their level of risk aversion. The agents live for one period and consumption takes place at the end of the period. There is a single risky asset in the economy, whose payoff at the end of the period is denoted by $\widetilde{x}$. We let p denote the unit price of the risky asset, which means that both agents can sell their property rights on the risky asset against the delivery of the sure quantity p at the end of the period. We assume that the agents have the same endowment, which consists of a half unit of the risky asset. As in the standard portfolio problem, agents determine the optimal composition of their portfolio, in other words their optimal exposure to the risk. The difference with the standard model stems from the fact that agents take into account their impact on prices and can manipulate their beliefs to take advantage of this impact. For example, an agent, who is risk tolerant, hence willing to be quite highly exposed to the risk, or equivalently interested in buying a high quantity of the risky asset, could act as if he believed that the asset was less interesting (or as if he were less interested in buying the asset) in order to benefit from a lower price.

We are interested in the characteristics of this economy at the (Nash) equilibrium. More precisely, we are interested in the following questions. Does this model of strategic beliefs lead to subjectivity in optimal beliefs? Does it generate heterogeneous beliefs? Is there a link between risk tolerance and optimal belief and what is the nature of this link? Is there a pessimistic/optimistic bias at the individual as well as at the collective level ? What are the consequences on the equilibrium characteristics and in particular on the risk premium ?

In this section, agents have CARA utility functions for consumption, more precisely, $u_1(c) = -\exp -\frac{c}{\theta_1}$ and $u_2(c) = -\exp -\frac{c}{\theta_2}$, where $\theta_i > 0$ denotes the degree of risk tolerance of agent i. Moreover, we assume that $\widetilde{x}$ is normally distributed, with mean μ and variance σ^2.

In the competitive Walrasian equilibrium model, an equilibrium price is such that agents reach optimal demands and markets clear. More precisely, the optimal demand $\alpha_i(p)$ of the risky asset that agent i will retain given price p maximizes the expected utility from trade

$$E\left[-\exp -\frac{\frac{1}{2}p + \alpha_i\left(\widetilde{x} - p\right)}{\theta_i}\right].$$

The optimal demand is then given by $\alpha_i(p) = \theta_i \frac{\mu - p}{\sigma^2}$. The market clearing condition $\alpha_1(p) + \alpha_2(p) = 1$ imposes then that the equilibrium price of the risky asset is given by $p^* = \mu - \frac{\sigma^2}{\theta_1 + \theta_2}$. Moreover, we obtain that the optimal demand at the equilibrium is given by $\alpha_i^* = \alpha_i(p^*) = \frac{\theta_i}{\theta_1 + \theta_2}$ so that the agent with a higher (resp. lower) risk tolerance level has a positive (resp. negative) demand in $\widetilde{x}$ and the part of the risk beared by agent i is exactly given by his relative level of risk tolerance.

As a simple extension of this model, we can consider the case in which agents have heterogeneous expectations about the payoff of the risky asset, i.e., agent 1 believes that $\widetilde{x}$ is normal with mean μ_1 and variance σ^2 and agent 2 believes that $\widetilde{x}$ is normal with mean μ_2 and variance σ^2 with[2] $\mu_1 \neq \mu_2$. The optimal demand of agent i given price p is then given by $\alpha_i(p, \mu_i) = \theta_i \frac{\mu_i - p}{\sigma^2}$. The market clearing price is given by

$$p(\mu_1, \mu_2) = \frac{\theta_1}{\theta_1 + \theta_2}\mu_1 + \frac{\theta_2}{\theta_1 + \theta_2}\mu_2 - \frac{\sigma^2}{\theta_1 + \theta_2}, \tag{1}$$

which is the equilibrium price in an economy in which agents share the same expectations given by $\frac{\theta_1\mu_1 + \theta_2\mu_2}{\theta_1 + \theta_2}$. In other words, the equilibrium price in an economy with heterogeneous beliefs is the equilibrium price in an economy in which the belief of the representative agent (whose risk tolerance $\overline{\theta}$ is given, as in the standard setting, by the sum of the individual risk tolerances) is given by the average of the heterogeneous beliefs, weighted by the risk tolerance. Moreover, we obtain that the optimal demand α_i^* of agent i at the equilibrium is given by

$$\alpha_i^*(\mu_1, \mu_2) = \alpha_i(p(\mu_1, \mu_2), \mu_i) = \frac{\theta_i}{\overline{\theta}}\left[1 + \theta_j \frac{\mu_i - \mu_j}{\sigma^2}\right] \tag{2}$$

and the part of the risk beared by agent i depends upon both his level of risk tolerance and his belief. Letting RP (resp. RP^{stdd}) denote the risk premium $\mu - p$ in this setting (resp. in the standard setting) we obtain

$$RP = \frac{\sigma^2}{\overline{\theta}} + \left(\mu - \frac{\theta_1\mu_1 + \theta_2\mu_2}{\theta_1 + \theta_2}\right) = RP^{stdd} + \left(\mu - \frac{\theta_1\mu_1 + \theta_2\mu_2}{\theta_1 + \theta_2}\right) \tag{3}$$

which means that the risk premium in an economy with heterogeneous subjective beliefs is higher than in the standard rational expectations setting if and only if the belief of the representative

[2] Walrasian equilibrium models with heterogeneous beliefs have been studied, among others, by Williams (1977), Abel (1989), Detemple-Murthy (1994), Calvet et al. (2002), Jouini-Napp (2006).

agent, which is the risk tolerance weighted average of the individual beliefs, is pessimistic, where pessimistic is meant in the sense that the mean of the risky asset's payoff is underestimated. In such a setting, it is particularly interesting to explore when and why individuals are pessimistic, as well as the nature of the link between risk tolerance and pessimism. In the present paper, the individual subjective beliefs are determined endogenously and we analyze their properties, especially in terms of pessimism, correlation between pessimism and risk tolerance and impact on the risk premium.

Instead of assuming that the equilibrium is a competitive one, we consider agents who act as imperfect competitors and take into account explicitly the effect their trading has on price. More precisely, as in Kyle (1989) in a setting with asymmetric information, each agent chooses a demand schedule to maximize his utility from profits, taking into account his effect on prices by considering as given the strategy the other agent uses to choose his demand schedule. This leads to the notion of Nash equilibrium in demand schedules[3]. We shall consider demand schedules of a specific form. We assume that the strategic variable for the agents consists in their belief about the expected payoff of the risky asset, i.e. the mean of the random variable $\widetilde{x}$. By changing his strategic belief on the mean of $\widetilde{x}$, the agent changes both the quantity he trades and the market clearing price at which he trades that quantity. Indeed, as we have seen above, the equilibrium price in a setting with heterogeneous beliefs and the optimal trading quantities are given by Equations (1) and (2).

The choice of a belief $\widehat{\mu}_i$ taking the belief μ_j of agent j as given is then determined by the maximization of the utility level

$$U_i\left(\mu_i\right) = E\left[u_i\left\{\frac{1}{2}p\left(\mu_i,\mu_j\right) + \left\{\widetilde{x} - p\left(\mu_i,\mu_j\right)\right\}\alpha_i^*\left(\mu_i,\mu_j\right)\right\}\right].$$

We emphasize that the choice of a given belief $\widehat{\mu}$ is strategic: as in a rational expectations equilibrium, the agent knows that the true mean of $\widetilde{x}$ is μ but he behaves as if he truly believed that it is $\widehat{\mu}$ in order to take advantage of his impact on prices and to maximize his utility from profits. Next section will provide situations where the choice of the beliefs as a strategic variable is natural.

[3]As underlined by Kyle "(this) is perhaps the most obvious modification of the conventional competitive rational expectations concept. It preserves market clearing through a Walrasian mechanism and keeps the Nash flavour of a competitive equilibrium." In fact, the concept of Nash equilibrium in demand schedules is the analogon, from the consumers point of view, of the Cournot-Nash equilibrium in supply for producers.

Definition 1 *A Nash equilibrium in beliefs (on the mean) is defined as a pair of strategies on the mean $M = (\widehat{\mu}_1, \widehat{\mu}_2)$ such that for any other pair of strategies M' differing only in the i-th component, for $i = 1, 2$, the strategy M yields a utility level no less than M' :*

$$E\left[u_i\left(\frac{1}{2}p(M) + (\widetilde{x} - p(M))\,\alpha_i^*(M)\right)\right] \geq E\left[u_i\left(\frac{1}{2}p(M') + (\widetilde{x} - p(M'))\,\alpha_i^*(M')\right)\right].$$

The construction of endogenous subjective beliefs that are solutions of a given utility maximization problem has been considered in recent literature by Brunnermeier-Parker (2005), Gollier (2005), Gollier-Muerman (2006). In our framework, the subjective beliefs are not only optimal but *strategic.* Indeed, they do not result from an individual utility maximization problem but from a Nash equilibrium, in which each agent takes into account the impact of his choices on the equilibrium price and allocations. In a non-strategic setting where agents choose their belief in order to maximize a criterion related to their well-being, it is immediate that the *optimal* belief must be optimistic for all agents and that all agents select a riskier portfolio. In our setting, there is no such immediate intuition for a given systematic bias. A more detailed comparison between the *optimal* beliefs approach and the *strategic* beliefs approach is provided in Section 4.

Proposition 1 *In a model with two agents, exponential utility functions and normal distributions, there exists a unique Nash equilibrium in beliefs (on the mean). It satisfies the following properties.*

1. *The optimal beliefs $M = (\widehat{\mu}_1, \widehat{\mu}_2)$ are given by*

$$\widehat{\mu}_1 = \mu - \frac{\sigma^2}{4\theta_2(\theta_1 + \theta_2)}(\theta_1 - \theta_2), \qquad \widehat{\mu}_2 = \mu - \frac{\sigma^2}{4\theta_1(\theta_1 + \theta_2)}(\theta_2 - \theta_1). \tag{4}$$

The more risk tolerant agent is pessimistic, in the sense that he behaves as if the mean of $\widetilde{x}$ lied below its true value, and the less risk tolerant agent is optimistic. Moreover, the more risk tolerant agent is more pessimistic than the less risk tolerant agent is optimistic and the unweighted average of the beliefs is pessimistic:

$$\frac{\widehat{\mu}_1 + \widehat{\mu}_2}{2} = \mu - \frac{1}{8}\frac{(\theta_1 - \theta_2)^2\sigma^2}{\theta_1\theta_2\overline{\theta}}. \tag{5}$$

2. *The representative agent is pessimistic, i.e. the average of the individual beliefs weighted*

by the risk tolerance is pessimistic. More precisely,

$$\frac{\theta_1\widehat{\mu}_1 + \theta_2\widehat{\mu}_2}{\theta_1 + \theta_2} = \mu - \frac{1}{4}\frac{(\theta_1 - \theta_2)^2 \sigma^2}{\theta_1\theta_2 (\theta_1 + \theta_2)} \tag{6}$$

3. *The risk premium RP (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely*

$$RP = RP^{stdd} + \left(\mu - \frac{\theta_1\widehat{\mu}_1 + \theta_2\widehat{\mu}_2}{\theta_1 + \theta_2}\right) = RP^{stdd} + \frac{1}{4}\frac{(\theta_1 - \theta_2)^2 \sigma^2}{\theta_1\theta_2 (\theta_1 + \theta_2)}.$$

4. *The optimal demands are given by*

$$\alpha_1^* = \frac{\theta_1}{\theta_1 + \theta_2} + \frac{(\theta_2 - \theta_1)}{4(\theta_1 + \theta_2)}, \qquad \alpha_2^* = \frac{\theta_2}{\theta_1 + \theta_2} + \frac{(\theta_1 - \theta_2)}{4(\theta_1 + \theta_2)}$$

   which means that the volumes of trade (and the risk sharing) are reduced compared to the standard setting. The more risk tolerant (resp. risk averse) agent selects a less (resp. more) risky portfolio.

5. *If $\frac{2}{3}\theta_1 \leq \theta_2 \leq \frac{3}{2}\theta_1$, then, at the equilibrium, both agents have utility levels that are lower than in the Walrasian setting. Otherwise, the utility level of the more risk tolerant agent increases (with respect to the Walrasian setting) while the utility level of the more risk averse agent decreases.*

Note first that our construction of strategic beliefs leads to subjective and heterogeneous optimal beliefs. Indeed, optimal strategic beliefs differ from the objective belief, agents 1 and 2 differ in their optimal belief and, as expressed in Equations (4), beliefs heterogeneity takes its roots in the difference in risk aversion levels. Besides, more than just being "heterogeneous", optimal beliefs are "antagonistic" in the sense that one of the agents is optimistic ($\widehat{\mu}_i > \mu$) and the other one is pessimistic ($\widehat{\mu}_i < \mu$).

The pessimism of the more risk tolerant agent can be interpreted as follows. Suppose that agent 1 is more risk tolerant. At the equilibrium, since agents initially only differ in their level of risk aversion, the risky asset's demand for agent 1 is positive. His expected utility from trade is then decreasing in the price of the risky asset. The choice of a pessimistic belief is associated to a lower demand, hence to a lower price and a higher expected utility. The optimal belief balances this benefit of pessimism against the costs of worse decision making. The converse reasoning applies to agent 2, who, at the equilibrium, has a negative demand in the risky asset and benefits from optimism. Another way to interpret the pessimism of the more risk tolerant agent is to analyze the situation in the neighborhood of the objective belief and the associated equilibrium, i.e. the Walrasian equilibrium. Indeed, a deviation from the objective belief has potentially two effects on the utility level: a quantity effect and a price effect. The quantity effect is equal to zero due to the optimal quantity choice condition in the Walrasian equilibrium and the price effect is positive for the less risk tolerant agent (i.e. the agent that has a negative net demand) and is negative for the more risk tolerant agent.

As a consequence of the positive correlation between pessimism (optimism) and risk tolerance (risk aversion), the more risk tolerant will insure the less risk tolerant less than in the standard setting, which induces less risk-sharing.

The average (unweighted) belief is pessimistic, which means that the risk tolerant agent is more pessimistic than the risk averse is optimistic. This result can be understood as follows. As we have seen, the optimal belief results from an arbitrage between the benefit of a low price induced by pessimism for the risk tolerant (resp. the benefit of a high price induced by optimism for the risk averse) against the costs of worse decision making. Let us explore this point further. At the Walrasian equilibrium, the marginal utility of the less risk tolerant agent associated to a marginal increase of μ is positive and equal to the marginal utility of the more risk tolerant agent associated to a marginal decrease of μ. By definition, the optimal beliefs correspond to zero marginal utilities. When the more risk tolerant agent becomes more pessimistic and when the less risk tolerant agent becomes more optimistic, their marginal utilities decrease at different rates. The difference between these two rates originates in the variance terms in the utility functions and more precisely in the terms $(\Delta\alpha_1)^2$ and $(\Delta\alpha_2)^2$. Due to the market clearing condition, these two terms are equal; however, by definition of the risk tolerance coefficient, they are weighted respectively by $\frac{1}{\theta_1^2}$ and $\frac{1}{\theta_2^2}$ in the utility functions. This leads to a slower decrease of the marginal utility for the more risk tolerant agent and then to a more pronounced divergence from the objective belief for that agent.

The consensus belief, which is given by the average of the individual beliefs weighted by the risk tolerance, is then obviously pessimistic. Intuitively, the more risk tolerant agents make the market, and the consensus belief reflects the characteristics of the more risk tolerant. Since we have just seen that the more risk tolerant is pessimistic, it is consistent to obtain a pessimistic consensus belief.

The risk premium is greater than in the standard rational expectations equilibrium, which is interesting in light of the risk premium puzzle. This is easily understandable, since, as we have seen, in equilibrium models with heterogeneous beliefs the risk premium is higher than in the standard setting if and only if the consensus belief is pessimistic. The reason why pessimism increases the market price of risk is not that a pessimistic representative agent requires a higher market price of risk. He requires the same market price of risk but his pessimism leads him to underestimate the average rate of return of the risky asset. Thus the objective expectation of the equilibrium market price of risk is greater than the representative agent's subjective expectation, hence is greater than the standard market price of risk (see Abel, 2002, and Jouini-Napp, 2006).

To sum up, our construction of endogenous beliefs leads to optimal beliefs that are different from the objective belief, heterogeneous, and antagonistic (one is optimistic and the other is pessimistic). There is a positive correlation between risk tolerance (resp. risk aversion) and pessimism (resp. optimism), which leads to less risk-sharing. The consensus belief is pessimistic, which leads to a higher equilibrium risk premium.

Our results are robust to variations in the initial endowments as long as the more risk tolerant agent has a positive net demand, i.e. as long as the more risk tolerant agent insures the less risk tolerant one, which is a natural situation. However, note that all the effects we exhibited disappear when there is no aggregate risk (i.e. when the total supply in risky assets is equal to zero). Indeed, in such a framework there is no trade at the Walrasian equilibrium and there is then no price effect and no utility gain associated to a deviation from the objective belief.

3 N-agents models and applications

The previous section focused on a 2 agents model in order to enlighten the impact of a strategic behavior on equilibrium prices and risk-premium. In this section, we generalize previous results to models with N agents. We first consider the case of small N for which the framework and the results appear as a direct generalization of those of the section above. In a second step, we consider a model with a large number of agents but with a small number of influential investors or gurus. In such a setting, gurus choose their beliefs strategically and the other agents adopt one of the gurus announced beliefs. For credibility reasons, gurus act according to their announced beliefs. Hence, even if their beliefs are not true beliefs, they will be considered as such by an econometrician/observer who analyzes portfolio choices. As far as the other agents are concerned, they sincerely trust one of the gurus and they adopt his announced beliefs as their own true beliefs. In such a model, restricting demand schedules manipulations to beliefs manipulations is natural. Indeed, gurus modify their beliefs in order to manipulate others beliefs. They have no specific reason nor incentives to modify other parameters of their demand schedules like the risk tolerance parameter or more generally the form of their utility function.

3.1 The case of a finite number of agents

Let us consider a model a model with N agents, indexed by $i = 1, ..., N$. Each agent is endowed with $\frac{1}{N}$ unit of the risky asset and we denote by θ_i the level of risk tolerance of agent i. We denote by $\overline{\theta}$ the risk tolerance of the representative agent, i.e. $\overline{\theta} = \sum_{i=1}^{N} \theta_i$.

If agent i chooses a belief μ_i, then his optimal trading quantity is given by $\alpha_i(p, \mu_i) = \theta_i \frac{\mu_i - p}{\sigma^2}$ where p denotes the market clearing price. The market clearing condition imposes then

$$p\left((\mu_i)_{i=1,...,N}\right) = \sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \mu_i - \frac{\sigma^2}{\overline{\theta}}, \tag{7}$$

which is the price in an economy with a representative agent (with risk-tolerance $\overline{\theta}$), whose belief is given by the average of the individual beliefs weighted by the risk-tolerance. The choice of a belief $\widehat{\mu}_i$ is then determined by the maximization of the utility level

$$U_i(\mu_i) = E\left[u_i\left\{\frac{1}{2}p(\widehat{\mu}_1, ..., \mu_i, ..., \widehat{\mu}_N) + \{\widetilde{x} - p(\widehat{\mu}_1, ..., \mu_i, ..., \widehat{\mu}_N)\} \alpha_i^*(\widehat{\mu}_1, ..., \mu_i, ..., \widehat{\mu}_N)\right\}\right],$$

where $\alpha_i^*(\widehat{\mu}_1, ..., \mu_i, ..., \widehat{\mu}_N) = \alpha_i^*(p(\widehat{\mu}_1, ..., \mu_i, ..., \widehat{\mu}_N), \mu_i)$ and where $\left(\widehat{\mu}_j\right)_{j \neq i}$ denotes the beliefs chosen by the other agents, and are taken as given.

With this definition we obtain the following result that is the analog of Proposition 1.

Proposition 2 *There exists a unique Nash equilibrium in beliefs in the model with N agents. It satisfies the following properties.*

1. *The optimal beliefs are given by* $\widehat{\mu}_i = \mu + \frac{\overline{\theta}^2 - \sum_{i=1}^{N} \theta_i^2 - (N-1)\theta_i \overline{\theta}}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N} \theta_i^2\right)}$, *for* $i = 1, ..., N$.

2. *The unweighted average of the beliefs is pessimistic, i.e.* $\frac{1}{N}\sum_{i=1}^{N} \widehat{\mu}_i < \mu$. *More precisely, it is given by*

$$\frac{1}{N}\sum_{i=1}^{N} \widehat{\mu}_i = \mu - \frac{\left(N\sum_{i=1}^{N} \theta_i^2 - \overline{\theta}^2\right)\sigma^2}{N^2\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N} \theta_i^2\right)}.$$

3. *There is a positive correlation between pessimism and risk-tolerance, in other words the more risk-tolerant agents are more pessimistic.*

4. *The representative agent is pessimistic, i.e. the average of the individual beliefs weighted by the risk-tolerance is pessimistic. More precisely,*

$$\sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \widehat{\mu}_i = \mu - \frac{\left(N \sum_{i=1}^{N} \theta_i^2 - \overline{\theta}^2\right) \sigma^2}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N} \theta_i^2\right)}. \tag{8}$$

5. *The risk premium (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely*

$$RP = RP^{stdd} + \left(\mu - \sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \widehat{\mu}_i\right) = RP^{stdd} + \frac{\left(N \sum_{i=1}^{N} \theta_i^2 - \overline{\theta}^2\right) \sigma^2}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N} \theta_i^2\right)}.$$

The concept of Nash equilibrium in beliefs with N agents is the natural generalization of the concept with 2 agents.

The model with N agents leads then to optimal beliefs that are different from the objective belief, heterogeneous, and we still have both optimistic and pessimistic agents. The correlation between risk tolerance (resp. risk aversion) and pessimism (resp. optimism) is still positive. The consensus belief is pessimistic, which leads to a higher equilibrium risk premium.

There are two ways to analyze the impact of a growing number of agents on the equilibrium parameters. We may consider the situation where we add new agents that are similar to the initial ones or the situation where we consider a larger set of agents but maintaining the total risk tolerance $\overline{\theta}$ constant. In both cases, the optimal beliefs converge to the objective belief and the risk-premium converges to the standard risk-premium. This result is natural, since the impact of the individual belief on the equilibrium price decreases when the number of agents increases. When N becomes large, each agent becomes price-taker and has no benefit from a belief manipulation. In particular, if we consider an economy with a set of K agents characterized by their risk tolerances $(\theta_i)_{i=1,\cdots,K}$ and if we consider the economy in which agent i, $i = 1, \cdots, K$, is replaced by N agents each with a risk tolerance level $\frac{\theta_i}{N}$, then the characteristics of the strategic equilibrium in this economy with NK agents converge to the characteristics of the Walrasian equilibrium in the economy with K agents.

This means that our results are particularly relevant in situations where there are a small number of agents. This is the case for financial markets when the capital is highly concentrated or for assets that require a very high level of technicality. More generally, this is also the case in all situations in which a small number of risk averse agents have to choose the optimal exposure to a risk. For example, when an insurance company has to negociate an optimal retention rate with reinsurance companies or when entrepreneurs have to fix the optimal proportion of equity to retain for a given project. These problems lead to a bargaining situation in which each agent tries to manipulate the equilibrium price by announcing a belief that is potentially different from his true belief and acts accordingly. Indeed, as in a standard bargaining situation for a given good, the "buyer" will try to lower the price (by depreciating the good) while the "seller" will try to higher the price (by singing the praise of the good) and the resulting price will depend on the relative bargaining power of the two individuals.

In the financial markets setting, our findings consist in an increase of the market price of risk which is interesting in light of the risk-premium puzzle. In the insurance industry, our results lead to a situation where the more risk averse agent (the insured) is optimistic and the less risk averse agent (the insurer) is pessimistic. The average belief is pessimistic leading to a higher insurance premium, which might help to explain the purchase of vastly overpriced insurance in a range of situations (Cutler-Zeckhauser, 2004). In corporate finance, IPO's can be modeled as a decision for a risk averse entrepreneur to sell shares of his firm to more risk tolerant investors. The application of our results to such a setting leads to a pessimistic consensus belief. As a result, the firm is underpriced and the short run return is large, which is consistent with the empirical literature on IPO's (Ibbotson and Ritter, 1995).

3.2 Large number of agents: gurus and beliefs formation

As we have seen in the previous section, the impact of the strategic behavior vanishes when the number of agents goes to infinity. However, as underlined by Shefrin (2005, p116) "the real issue is not so much how much investors they are but to what extent investors cluster in their beliefs".

We assume that there is a continuum of infinitesimal agents with risk tolerance levels θ_i for $i \in [0,1]$, where $i \to \theta_i$ is measurable. We also assume that there are, among these agents, N influential investors who have a wide audience. We do not assume that they have the capacity to move the markets by their own trades but only that they have, through their wide audience, the ability to largely influence the other investors. We will call them gurus.

Gurus announce their beliefs and act accordingly in order to maintain their credibility. As in Benabou-Laroque (2001), we have in mind the "guru" who “issues forecasts or newsletters but is also in the business of trading for his own account or some investment firm”.

The other agents in the economy do not have specific information and believe that gurus are well informed or have specific ability to predict market movements. At a given date, based on past realizations and gurus’ recommendations, each agent has a preferred guru and adopts his beliefs. There are then N groups of agents: the agents in Group i (G_i) follow guru i, $i = 1, ..., N$.

Let us assume that $N = 2$ and let us denote by $\Theta_1 = \int_{G_1} \theta_i di$ and by $\Theta_2 = \int_{G_2} \theta_i di$ the aggregate risk tolerance in G_1 and G_2. We have $\Theta_1 + \Theta_2 = \Theta$ where Θ is the aggregate risk tolerance in the economy.

Guru j, $j = 1, 2$, announces a belief μ_j and this belief is adopted by all the agents in G_j. The agents in G_j can then be aggregated into a representative agent with belief μ_j and risk tolerance Θ_j. For a given price p for the risky asset, their aggregate demand is then given by $\Theta_j \frac{\mu_j - p}{\sigma^2}$.

There is one unit of the risky asset in the economy and it is uniformly distributed among the agents. The equilibrium price $p(\mu_1, \mu_2)$ is then determined by the condition

$$\Theta_1 \frac{\mu_1 - p}{\sigma^2} + \Theta_2 \frac{\mu_2 - p}{\sigma^2} = 1.$$

The demand of guru j is determined, as usual, by his own level of risk tolerance denoted and by θ_j^* and by his announced belief μ_j. Indeed, gurus’ demands are observable, hence they act according to their announced beliefs in order to maintain their credibility.

The demand of guru j given price p is then given by $\theta_j^* \frac{\mu_j - p}{\sigma^2}$ and his utility level at the equilibrium is given by

$$U_j = -\exp\left(-\theta_j^* \frac{\mu_j - p(\mu_1, \mu_2)}{\sigma^2} (\mu - p(\mu_1, \mu_2)) - p(\mu_1, \mu_2) + \frac{1}{2} \frac{1}{\theta_j^*} \left(\theta_j^* \frac{\mu_j - p(\mu_1, \mu_2)}{\sigma^2} \right)^2 \sigma^2 \right).$$

Each guru chooses then optimally his belief, the belief of the other guru being given. The two gurus participate then to a game characterized by payoff functions (U_1, U_2) with parameters $(\Theta_1, \Theta_2, \theta_1^*, \theta_2^*)$ and look for a Nash equilibrium (μ_1, μ_2).

The other agents are able to observe market realizations and to compare them with gurus’ predictions. In a repeated game and at a given stage of the game, they may choose the guru that seems to be more accurate on the basis of previous stages and move from one group to another accordingly. In a steady state, we should then have

$$|\mu_1 - \mu| = |\mu_2 - \mu| \,.$$

In the next, we will say that the game $(\Theta_1, \Theta_2, \theta_1^*, \theta_2^*)$ has a stable Nash equilibrium if we have

$$|\mu_1 - \mu| = |\mu_2 - \mu|$$

at the equilibrium.

Finally, when $|\mu_1 - \mu| = |\mu_2 - \mu|$, agents should be indifferent between guru 1 and guru 2. Indeed, there is no specific reason to choose one guru rather than the other one (or no reasons that are related to the model). They should then choose each of the two gurus with a probability $\frac{1}{2}$ and the two groups should be of equal size, i.e. $\int_{G_1} di = \int_{G_2} di = \frac{1}{2}$.

Proposition 3 *Let us consider a model with a continuum of agents and two gurus. We denote by $\theta_1^* > \theta_2^*$ the risk tolerances of guru 1 and guru 2.*

1. *The game $(\Theta_1, \Theta_2, \theta_1^*, \theta_2^*)$ has a unique Nash equilibrium (μ_1, μ_2).*
2. *If we assume that one guru is more risk tolerant than the average and that the other guru is less risk tolerant than the average, i.e. $\theta_1^* > \Theta > \theta_2^*$, then there exists a unique pair (Θ_1, Θ_2) such that $\Theta_1 + \Theta_2 = \Theta$ for which the Nash equilibrium is stable. This means that there is a unique distribution of risk tolerance among Group 1 and Group 2 for which the Nash equilibrium is stable.*
3. *Let us consider a stable Nash equilibrium. If $\Theta = \frac{\theta_1^* + \theta_2^*}{2}$ or in other words if the gurus are on average as risk tolerant as the agents in the economy then the more risk tolerant guru (agent 1) is the more pessimistic one. Furthermore, group 1 has a higher aggregate risk tolerance level, i.e. $\Theta_1 > \Theta_2$ and the more risk tolerant group is the more pessimistic one.*
4. *Let us consider a stable Nash equilibrium. If $\Theta = \frac{\theta_1^* + \theta_2^*}{2}$ and if the two groups are of equal size, then there is a positive correlation between pessimism and risk tolerance among the whole population.*

In other words, if the game we consider is part of a repeated game, then at a steady state we should observe a situation where guru 1 is pessimistic and guru 2 is optimistic and where the aggregate risk tolerances (Θ_1, Θ_2) of group 1 and group 2 are such that the optimal strategy for both gurus leads them to have the same level of accuracy with respect to the objective belief. Furthermore, both groups have the same size. The agents have then no specific reason to move from one group to another and even if they choose their group randomly this will not lead to a modification of (Θ_1, Θ_2). In such an equilibrium, there is a positive correlation between pessimism and risk tolerance among the whole population.

This means that the strategic behavior may explain a higher risk premium when agents' cluster in their beliefs. This clustering property is, in particular, documented by Fisher and Statman who claim that "there is a positive relationship between changes in the sentiment of individual investors and that of newsletter writers".

4 Extensions and generalizations

In this section, we analyze the robustness of our results (heterogeneity of optimal beliefs, aggregate pessimism, positive correlation between pessimism and risk tolerance, positive impact on the risk premium) to other specifications of the model. In particular, we consider extensions of the simple model of Section 2 in the following directions; a model in which the strategic variable is the variance (instead of the mean) of the payoff of the risky asset, a model with more general utility functions and distributions and a model with more than one risky asset.

4.1 Nash equilibrium in beliefs on the variance

In this section we consider a Nash equilibrium in beliefs where the agents differ by their estimation of the variance of the risky asset. The financial model is the same as in Section 2 except that the strategic variable is now the variance of $\widetilde{x}$. The payoff of the risky asset $\widetilde{x}$ is still normally distributed with mean μ and variance σ^2, but the agents can choose to behave as if the variance was σ_i^2 in order to maximize their utility from profits from trade. Here again, the agents know the true distribution of $\widetilde{x}$, and the choice of a different variance is only strategic.

In the competitive Walrasian equilibrium model, both agents agree on the true value of the variance of the payoff of the risky asset; as we have seen in Section 2, the optimal demand of agent i, given the equilibrium price p, is given by $\alpha_i(p) = \theta_i \frac{\mu - p}{\sigma^2}$ and the market clearing condition $\alpha_1(p) + \alpha_2(p) = 1$ imposes that the equilibrium price of the risky asset is given by $p = \mu - \frac{\sigma^2}{\theta_1 + \theta_2}$.

Let us first assume that agents have heterogeneous expectations about the payoff of the risky asset, i.e., agent 1 believes that $\widetilde{x}$ has variance σ_1^2 and agent 2 believes that $\widetilde{x}$ has variance σ_2^2 with $\sigma_1 \neq \sigma_2$. The bias with respect to the objective belief can here be interpreted as a form of doubt ($\sigma_i^2 > \sigma^2$) or overconfidence ($\sigma_i^2 < \sigma^2$) instead of the pessimism/optimism biases[4] of Section 2. The optimal demand of agent i in this setting is given by $\alpha_i(p, \sigma_i) = \theta_i \frac{\mu - p}{\sigma_i^2}$. It is then easy to obtain that the market clearing price is given by $p(\sigma_1, \sigma_2) = \mu - \left(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2}\right)^{-1}$. Note that this price is the equilibrium price in an economy in which agents share the same belief, namely the harmonic average of the initial beliefs, weighted by the risk tolerance. In other words, it is the equilibrium price in an economy in which the belief of the representative agent (whose risk tolerance is given by $\overline{\theta}$, as in the standard setting) is given by the average of the initial beliefs, weighted by the risk tolerance[5]. In this setting, the equilibrium risk premium is given by $\mu - p(\sigma_1, \sigma_2) = \left(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2}\right)^{-1}$, which means that the risk premium in an economy with heterogeneous subjective beliefs is higher than in the standard rational expectations setting if and only if the belief of the representative agent exhibits doubt (i.e., $(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2})^{-1} > \frac{\sigma^2}{\theta_1 + \theta_2}$). We now analyze the properties of the optimal beliefs in the context of a Nash equilibrium in beliefs on the variance.

We assume that the strategic variable for the agents consists in the variance of $\widetilde{x}$. The choice of a belief $\widehat{\sigma}_i$ is then determined by the maximization of the utility level

$$U_i(\sigma_i) = E\left[u_i\left\{\frac{1}{2}p(\sigma_i, \widehat{\sigma}_j) + \{\widetilde{x} - p(\sigma_i, \widehat{\sigma}_j)\}\, \alpha_i^*(\sigma_i, \widehat{\sigma}_j)\right\}\right].$$

where $\alpha_i^*(\sigma_i, \widehat{\sigma}_j) = \alpha_i(p(\sigma_i, \widehat{\sigma}_j), \sigma_i)$ and where $\widehat{\sigma}_j$, for $j \neq i$, is considered as given.

Definition 2. *A Nash equilibrium in beliefs on the variance is defined as a pair of variance strategies $M = (\widehat{\sigma}_1, \widehat{\sigma}_2)$ such that for any other pair of strategies M' differing only in the i-th component, for $i = 1, 2$, the strategy M yields a utility level no less than M' :*

$$E\left[u_i\left\{\frac{1}{2}p(M) + \{\widetilde{x} - p(M)\}\, \alpha_i^*(M)\right\}\right] \geq E\left[u_i\left\{\frac{1}{2}p(M') + \{\widetilde{x} - p(M')\}\, \alpha_i^*(M')\right\}\right].$$

Proposition 4 *There exists a unique Nash equilibrium in beliefs on the variance. It satisfies the following properties.*

[4] See Abel (2002) for concepts of pessimism and doubt related to first and second order stochastic dominance.

[5] Walrasian equilibrium models in which agents have heterogeneous beliefs on the variance of the asset under consideration have been studied by, among others, Abel (1989, 2002) and Jouini-Napp (2006).

1. *The optimal beliefs* $M = (\widehat{\sigma}_1, \widehat{\sigma}_2)$ *are given by*

$$\widehat{\sigma}_1^2 = \sigma^2 \left(1 + \frac{\theta_1 - \theta_2}{4\theta_2}\right), \qquad \widehat{\sigma}_2^2 = \sigma^2 \left(1 + \frac{\theta_2 - \theta_1}{4\theta_1}\right). \tag{9}$$

The more risk tolerant agent exhibits doubt, in the sense that he behaves as if the variance of $\widetilde{x}$ *lied above its true value, and the less risk tolerant agent is overconfident. Moreover, the more risk tolerant agent exhibits more doubt than the less risk tolerant agent exhibits overfidence, and the unweighted (harmonic) average of the beliefs exhibits doubt:*

$$2\left(\frac{1}{\widehat{\sigma}_1^2} + \frac{1}{\widehat{\sigma}_2^2}\right)^{-1} = \sigma^2 \left(1 + \frac{(\theta_1 - \theta_2)^2}{2\left(\theta_1^2 + \theta_2^2 + 6\theta_1\theta_2\right)}\right). \tag{10}$$

2. *The representative agent exhibits doubt, i.e. the harmonic average of the individual beliefs weighted by the risk tolerance exhibits doubt. More precisely,*

$$(\theta_1 + \theta_2)\left(\frac{\theta_1}{\widehat{\sigma}_1^2} + \frac{\theta_2}{\widehat{\sigma}_2^2}\right)^{-1} = \sigma^2 \left(1 + \frac{3\left(\theta_1 - \theta_2\right)^2}{16\theta_1\theta_2}\right).$$

3. *The risk premium (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely*

$$RP = RP^{stdd} + \left(\left(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2}\right)^{-1} - \frac{\sigma^2}{\theta_1 + \theta_2}\right) = RP^{stdd} + \frac{3\left(\theta_1 - \theta_2\right)^2}{16\theta_1\theta_2\left(\theta_1 + \theta_2\right)}\sigma^2.$$

The Nash equilibrium in beliefs on the variance has then the same properties as the Nash equilibrium in beliefs on the mean except that pessimism is replaced by doubt. Note that behavioral studies in a non strategic context generally exhibit overconfidence instead of doubt. This bias is largely documented in the behavioral literature and in particular by Shiller (2000, p. 142): "Yet some basic tendency towards overconfidence appears to be a robust human character trait: the bias is definitely toward overconfidence rather than underconfidence". The strategic framework induces then an effect in the opposite direction and this might explain that Giordani and Söderlind (2006) "find little evidence of either overconfidence or doubt" in the survey of professional forecasters. Indeed, professional forecasters know that their forecasts have a direct as well as an indirect (their predictions influence the beliefs of many other investors) impact on prices and it is then natural for them to adopt a strategic behavior.

4.2 More general utility functions and distributions

The purpose of this section is to analyze the robustness of the results of Section 2 to more general utility functions and distributions. We consider a family of beliefs $\left(P_{\tilde{x}}^{\mu}\right)_{\mu \in K}$, corresponding to the possible subjective distributions for $\tilde{x}$, where $K \subset \mathbb{R}_+$ is a given set of admissible beliefs (including the objective one). For $\mu \in K$, we let $f(., \mu)$ denote the density function of $P_{\tilde{x}}^{\mu}$ with respect to the Lebesgue measure on $\mathbb{R}_+$.

As in Section 2, our economy is composed of two agents, initially endowed with a half unit of the risky asset $\tilde{x}$, who can manipulate their beliefs and choose an optimal composition of their portfolio, taking into account the effect their trading has on price.

We make the following assumptions

Assumption (A)

- The utility functions u_1 and u_2 are increasing, strictly concave and twice continuously differentiable on $\mathbb{R}_+$,
- Inada conditions: $u_i'(0) = +\infty$ and $\lim_{x \to \infty} u_i'(x) = 0$,
- The family $\left(P_{\tilde{x}}^{\mu}\right)_{\mu \in K}$ is increasing in the sense of the first stochastic dominance, i.e. $\int_0^x f(s, \mu) ds \geq \int_0^x f(s, \mu') ds$ for $\mu' \geq \mu$ in K.
- The functions $s \mapsto s u_i'(s)$, $i = 1, 2$, are increasing.

The first condition is standard. The second one guarantees interior solutions to the individual portfolio choice problem. The third condition ensures an order on the set of admissible beliefs. The setting of Section 2 satisfies this condition. More generally, any family of beliefs $\left(P_{\tilde{x}}^{\mu}\right)_{\mu \in \mathbb{R}_+}$ such that $f(s, \mu) = g(s - \mu)$ for a given distribution function g on $\mathbb{R}_+$ satisfies this monotonicity condition. Another example is provided by a family of log-normal distributions, $\left(\ln \mathcal{N}(\mu, \sigma^2)\right)_{\mu \in \mathbb{R}}$. The last condition guarantees that a first stochastic dominance shift in payoffs increases the demand for the risky asset (see Gollier, 2001). The same property can be obtained without this condition if we replace the first-stochastic dominance by the monotone likelihood ratio order (Landsberger and Meilijson, 1990).

In the next, we also assume that all the considered expectations exist and are finite. We may assume without any loss of generality that the objective distribution corresponds to $\mu = 0$ and we will simply denote by E (instead of E^0) the associated expectation operator[6].

[6] We let E^{μ} denote the expectation operator under the density $f(., \mu)$, i.e. $E^{\mu}\left[g(\tilde{x})\right] = \int g(s) f(s, \mu) dx$.

As in Section 2, agents choose their beliefs which means here that they choose a given μ and its associated subjective distribution $P_{\widetilde{x}}^{\mu}$ for $\widetilde{x}$.

For a given belief $P_{\widetilde{x}}^{\mu}$, the demand function of agent i is given by

$$\alpha_i(p,\mu) = \arg\max_{\alpha_i} E^{\mu}\left[u_i(\alpha_i(\tilde{x}-p)+\frac{1}{2}p)\right].$$

For a pair of beliefs (μ_1,μ_2), the equilibrium price $p(\mu_1,\mu_2)$ is determined by the market-clearing condition $\alpha_1(p(\mu_1,\mu_2),\mu_1)+\alpha_2(p(\mu_1,\mu_2),\mu_2)=1$ and the associated optimal demand for agent i is defined by $\alpha_i^*(\mu_1,\mu_2)=\alpha_i(p\left(\mu_1,\mu_2\right),\mu_i)$. Finally, the optimal belief $\widehat{\mu_i}$ of agent i is determined, as in Section 2, by the following optimization program

$$\arg\max_{\mu_i} E\left[u_i(\alpha_i^*(\mu_1,\mu_2)(\tilde{x}-p\left(\mu_1,\mu_2\right))+\frac{1}{2}p\left(\mu_1,\mu_2\right))\right]. \tag{11}$$

If this problem admits an interior solution $\widehat{\mu_i}$ in K, this solution satisfies the following first order condition

$$E\left[\left(\frac{d\alpha_i^*}{d\mu_i}(\tilde{x}-p)+(\frac{1}{2}-\alpha_i^*)\frac{dp}{d\mu_i}\right)u_i'(\alpha_i^*(\tilde{x}-p)+\frac{1}{2}p)\right]=0. \tag{12}$$

Proposition 5 *Under Assumption (A), the functions $\alpha_i(p,\mu)$, $p(\mu_1,\mu_2)$ and $\alpha_i^*(\mu_1,\mu_2)$ are well defined and satisfy $\frac{\partial\alpha_i}{\partial p}(p,\mu)\leq 0$, $\frac{\partial\alpha_i}{\partial\mu}(p,\mu)\geq 0$, $\frac{\partial p}{\partial\mu_i}\geq 0$, $\frac{\partial\alpha_i^*}{\partial\mu_i}\geq 0$, $\frac{\partial\alpha_i^*}{\partial\mu_j}\leq 0$, $i=1,2$, $j\neq i$.*

If the optimization program (11) admits an interior solution, then one of the agents (agent i) is pessimistic and the other one (agent j) is optimistic and we have $\alpha_j^*(\mu_1^*,\mu_2^*)\leq\frac{1}{2}\leq\alpha_i^*(\mu_1^*,\mu_2^*)$.

If one of the utility functions (say u_1) is more risk averse than the other one in the sense of Arrow-Pratt, then $\alpha_1^*(\mu_1,\mu_2)\leq\frac{1}{2}$, hence there is a positive correlation between pessimism and risk tolerance.

We obtain first that the optimal demand of the agents (as a function of the price and the belief) increases with the belief and decreases with the price, which are natural properties. As a consequence, the equilibrium price increases with the beliefs, which is also natural; if the asset is more "desirable", its equilibrium price increases. An increase in the belief of agent i has then two effects on his demand α_i^* at the equilibrium, a direct positive effect and an indirect negative effect due to the price increase. The global effect is positive. The effect of an increase of the belief of agent i on the equilibrium demand α_j^* of the other agent is negative because there is only one effect, namely the price effect.

We obtain that heterogeneity of optimal beliefs is robust to the choice of more general utility functions and distributions. Moreover, as in Section 2, one agent is optimistic and the other agent is pessimistic. The pessimistic agent is the one for which the net demand is positive. This result can be explained as before. For the agent who expresses a positive net demand for the risky asset, the choice of a pessimistic belief is associated to a lower price and a higher expected utility; the optimal belief balances this benefit of pessimism against the costs of worse decision making. The converse reasoning applies to the other agent, who, at the equilibrium, has a negative net demand in the risky asset and benefits from optimism.

The positive correlation between pessimism and risk tolerance is also robust to this more general setting. When one of the agents is more risk tolerant, his net demand is necessarily positive. Otherwise, he would have a negative demand which would lead to an optimistic belief while the other agent would be pessimistic, more risk averse with a positive net demand. This is obviously impossible. The positive correlation follows.

4.3 The model with two risky assets

The model is essentially the same as in Section 2 except that we now suppose that there are two risky assets in the economy, whose associated payoffs at the end of the period are respectively denoted by $\widetilde{x}$ and $\widetilde{y}$. We let p (resp. q) denote the price of $\widetilde{x}$ (resp. $\widetilde{y}$) and we assume that $\widetilde{x}$ and $\widetilde{y}$ are normally distributed, more precisely $\widetilde{x} \sim \mathcal{N}\left(\mu, \sigma^2\right)$ and $\widetilde{y} \sim \mathcal{N}\left(\nu, \varpi^2\right)$. We let ρ denote the correlation between $\widetilde{x}$ and $\widetilde{y}$, i.e., $\rho \equiv \frac{cov(\widetilde{x},\widetilde{y})}{\sigma\varpi}$. Each agent is initially endowed with one half unit of each risky asset.

We assume that each agent can choose a belief, i.e. a pair (μ_i, ν_i) that maximizes his utility from trade and as previously we look for a Nash equilibrum in beliefs (on the means). The definition and the notations are straightforward generalizations of those introduced in Section 2.

Proposition 6 *There exists a unique Nash equilibrium in beliefs. It satisfies the following properties.*

1. *The optimal beliefs $M = ((\widehat{\mu}_i, \widehat{\nu}_i)\,; i = 1, 2)$ are given by*

$$\widehat{\mu}_i = \mu - \frac{(\theta_i - \theta_j)\left(\sigma^2 + \sigma\varpi\rho\right)}{4\theta_j\overline{\theta}}, \qquad \widehat{\nu}_i = \nu - \frac{(\theta_i - \theta_j)\left(\varpi^2 + \sigma\varpi\rho\right)}{4\theta_j\overline{\theta}}.$$

2. The optimal demands for agent i are given by

$$\alpha_i = \frac{1}{4}\frac{\varpi(\theta_i - \theta_j) + \sigma(\theta_j + 3\theta_i + 4\rho\theta_i)}{(\rho+1)\,\sigma\overline{\theta}}, \qquad \beta_i = \frac{1}{4}\frac{\sigma(\theta_i - \theta_j) + \varpi(\theta_j + 3\theta_i + 4\rho\theta_i)}{(\rho+1)\,\varpi\overline{\theta}}.$$

3. The risk premium (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely

$$\begin{aligned} \mu - p &= RP^{stdd}(\widetilde{x}) + \frac{1}{4}\frac{(\theta_1 - \theta_2)^2\left(\sigma^2 + \sigma\varpi\rho\right)}{\theta_1\theta_2\overline{\theta}} \\ \nu - q &= RP^{stdd}(\widetilde{y}) + \frac{1}{4}\frac{(\theta_1 - \theta_2)^2\left(\varpi^2 + \sigma\varpi\rho\right)}{\theta_1\theta_2\overline{\theta}} \end{aligned}$$

where $RP^{stdd}(\widetilde{x})$ and $RP^{stdd}(\widetilde{y})$ denote the standard risk-premium for $\widetilde{x}$ and $\widetilde{y}$ in an homogenous beliefs setting.

As far as the market portfolio is concerned, the market risk-premium RP^M and the beliefs ξ_i^M on the average market return are given by

$$\begin{aligned} \xi_i^M &= \xi - \frac{(\theta_i - \theta_j)\,\sigma_M^2}{4\theta_j\overline{\theta}} \\ RP^M &= RP^{stdd}(\widetilde{x}) + \frac{1}{4}\frac{(\theta_1 - \theta_2)^2\,\sigma_M^2}{\theta_1\theta_2\overline{\theta}} \end{aligned}$$

where $\xi = \mu + \nu$ and $\sigma_M^2 = \varpi^2 + 2\rho\sigma\varpi + \sigma^2$ correspond respectively to the objective market portfolio return and variance. These formulas are exactly the same as in the one asset framework which means that the more risk tolerant (risk averse) agent is pessimistic (resp. optimistic) at the aggregate level and the consensus belief is pessimistic at the aggregate level. The formulas for individual assets that are provided in the proposition are similar to those obtained in the one asset framework. However, for each asset, the variance term in the one-asset formula is replaced by the covariance of the considered asset payoffs with the market portfolio payoffs. Recall that in the Walrasian setting (CAPM setting), the equilibrium price for a given asset depends on the covariance of the payoffs of this asset with the payoffs of the market portfolio and not on the total variance of the asset payoffs. Since the agents modify their beliefs in order to manipulate the prices, it is natural to obtain optimal beliefs that depend on the covariance with the market portfolio and not on the total variance. The aggregate level properties (pessimism, correlation between pessimism and risk tolerance,...) are then retrieved at the individual assets level as far as these assets are positively correlated with the market portfolio.

As in the one risky asset framework, the strategic behavior leads to less risk-sharing than the Walrasian setting. It is interesting to note that this effect is more pronounced for the riskier asset. Intuitively, the strategic behavior leads to more beliefs dispersion for the riskier asset and hence to a more pronounced impact on the market for the riskier asset .

5 Strategic vs "optimal" beliefs

Let us compare our results with those that are obtained in an *optimal* non-strategic framework. More precisely, adopting the same framework and notations as in Section 2, we consider the following concept of optimal beliefs, which corresponds to a simplified version of Brunnermeier-Parker (2005) and Gollier (2005).

Definition 3. *For a given price p, an optimal (non-strategic) belief $\overline{\mu}_i(p)$ for agent i is defined as the solution of*

$$\arg\max_{\mu_i \in K} E_i \left[u_i \left(\frac{1}{2}p + \alpha_i(p, \mu_i)\left(\widetilde{x} - p\right) \right) \right]$$

where E_i is the expectation operator associated[7] *to the belief μ_i and where K is a given set of admissible values for μ_i that contains the objective belief μ.*

The belief $\overline{\mu}_i(p)$ is optimal in the sense that it maximizes over the set K the well-being of agent i. We can then define an associated equilibrium concept as follows.

Definition 4. *An equilibrium price with optimal (non-strategic) beliefs is defined as a price $\overline{p}$ such that agents have optimal demands and optimal (non-strategic) beliefs and such that markets clear, i.e.*

$$\alpha_1(\overline{p}, \overline{\mu}_1(\overline{p})) + \alpha_2(\overline{p}, \overline{\mu}_2(\overline{p})) = 1.$$

Let us assume that $K = [a, b]$.

Proposition 7 *In the setting of the exponential utility and normal distributions, we have*

1. *For a given price p, the optimal (non-strategic) belief $\overline{\mu}_i(p)$ for agent i, solves*

$$\max_{\mu \in \{a,b\}} (\mu - p)^2.$$

2. *If $\frac{\sigma^2}{\theta_1+\theta_2} \geq \frac{b-a}{2}$, then the equilibrium is characterized by $\overline{p} = b - \frac{\sigma^2}{\theta_1+\theta_2}$ and $\overline{\mu}_1(\overline{p}) = \overline{\mu}_2(\overline{p}) =$*

[7] More precisely, E_i is the expectation operator associated to a probability P_i that represents agents i's belief and under which $\widetilde{x} \sim \mathcal{N}(\mu_i, \sigma^2)$.

b. The agents share the same optimistic belief and the risk premium is lower than in the standard setting.

3. *If* $\frac{\theta_1 a+\theta_2 b}{\theta_1+\theta_2} - \frac{\sigma^2}{\theta_1+\theta_2} = \frac{a+b}{2}$*, where* $\theta_1 < \theta_2$*, then the equilibrium is characterized by* $\overline{p} = \frac{a+b}{2}$*,* $\overline{\mu}_1(\overline{p}) = a$ *and* $\overline{\mu}_2(\overline{p}) = b$*. The more risk tolerant agent is the more optimistic and the consensus belief* $\frac{\theta_1 a+\theta_2 b}{\theta_1+\theta_2}$ *is more optimistic than the equally weighted belief* $\frac{a+b}{2}$*. If the set of admissible beliefs is symmetric with respect to the objective belief* μ *(i.e.* $\mu = \frac{a+b}{2}$*), then the consensus belief is optimistic and the risk-premium is lower than in the standard setting.*

Notice that for $\frac{\sigma^2}{\theta_1+\theta_2} < \frac{b-a}{2}$ and $\frac{\sigma^2}{\theta_1+\theta_2} \neq \frac{a+b}{2} - \frac{\theta_1 a+\theta_2 b}{\theta_1+\theta_2}$ where agent 1 is the less risk tolerant one, there is no equilibrium. This is no more true if we consider the natural extension of our model to a model with a continuum of agents. It suffices then to assume that a proportion α of these agents choose the belief a and a proportion $(1-\alpha)$ choose the belief b. If we assume that the distribution of beliefs is independent of the distribution of risk tolerances, the market clearing condition leads to

$$\alpha a + (1-\alpha)\, b - \frac{\sigma^2}{\int \theta_i di} = \frac{a+b}{2}. \tag{13}$$

The proportion α is then perfectly determined if $\frac{\sigma^2}{\int \theta_i di} \leq \frac{b-a}{2}$. The solution α is always lower than $\frac{1}{2}$ which means that the consensus belief is always optimistic. This equilibrium in which each agent is indifferent between two possible beliefs and in which the market clearing condition imposes the proportions of agents choosing each belief resembles the equilibrium obtained in Brunnermeier-Parker (2005).

In fact, when the agents are not strategic and choose "optimal" beliefs, all agents are "optimistic about their equilibrium allocation", i.e. overestimate the return of their portfolio. Indeed, agents that are long in the risky asset choose a belief b and those that are short in the risky asset choose a belief a leading them to overestimate the return of their portfolio. This bias is induced by the way beliefs are constructed.

These results are analogous to those of Brunnermeier-Parker (2005) even if in their case there is no aggregate risk[8] and the function to be maximized is an average of the objectively expected utility and the subjectively expected utility. It is easy to check that we would obtain the same kind of results if we considered a weighted average of the objectively expected utility

[8] In this case, there is no absolute concept of optimism or pessimism and both agents are optimistic with respect to their own equilibrium allocation.

and the subjectively expected utility as in Gollier (2005)

$$\max_{\mu_i \in K} \left\{ \beta E\left[u_i\left(\frac{1}{2}p + \alpha_i(p,\mu_i)\left(\widetilde{x}-p\right) \right)\right] + (1-\beta)E_i\left[u_i\left(\frac{1}{2}p + \alpha_i(p,\mu_i)\left(\widetilde{x}-p\right) \right)\right] \right\}.$$

For β large enough, in other words when the weight on the objective expectation is beyond a given threshold, then agents share the same belief and this belief is optimistic. Otherwise, there is not a unique optimal belief, agents have extreme beliefs (i.e. a or b), but the possible equilibria still lead to an optimistic average belief[9]. In all cases, the average optimal belief is optimistic leading to a lower risk premium. These results are similar to those obtained by Gollier (2005) in a general discrete distributions setting.

To conclude, in the optimal (non-strategic) setting, the consensus belief is always optimistic and the risk premium is always lower than in the rational expectations setting. Furthermore, except for specific degenerate situations (see Equation 13), the agents share the same belief that is optimistic. There is then no beliefs heterogeneity induced by risk tolerance heterogeneity.

The difference between *optimal* (non strategic) and strategic beliefs is now clear, since in the latter setting, there is beliefs heterogeneity, one agent is optimistic while the other is pessimistic and the consensus belief is pessimistic.

6 Conclusion

The introduction of strategic interaction in the standard portfolio/equilibrium model provides a rationale for beliefs heterogeneity; it leads to optimal beliefs that are subjective, heterogeneous and antagonistic. The selection of optimal beliefs is governed by very precise rules. First, these beliefs must be related to the individual level of risk aversion: the beliefs of more risk averse agents exhibit optimism and/or overconfidence and the beliefs of more risk tolerant agents exhibit pessimism and/or doubt. As a consequence, there is a positive correlation between pessimism/doubt and risk tolerance. Second, the average belief exhibits pessimism and/or doubt as well as the belief of the representative agent. This is compatible with the observation

[9] More precisely, for $\beta < \frac{1}{2}$, the agents have extreme beliefs, a or b, as above and there might exist equilibria with heterogeneous optimal beliefs if the model parameters satisfy a condition like condition (13). For $\beta > \frac{1}{2}$, i.e. when there is more weight on the objective expectation, and if b is sufficiently large ($b > b^*$ for some b^*) the agents share the same belief and this belief is an interior point of $[a, b]$. For $\beta = \frac{1}{2}$ or for $b \leq b^*$ the agents share the same belief b.

that subjects in experimental and empirical studies exhibit a dose of pessimism (Wakker, 2001, Ben Mansour et al., 2006, Giordani-Söderlind, 2006). The experimental studies in a non-strategic framework generally conclude to the presence of overconfidence. However there is little evidence of overconfidence or doubt in financial markets empirical studies (Giordani-Söderlind, 2006). This might be explained by a doubt effect due to the strategic behavior that cancels the overconfidence bias. This induced pessimism/doubt of investors might be helpful to solve the equity premium puzzle. It is also helpful to explain the purchase of vastly overpriced insurance contracts or the large short run returns of IPOs.

This work suggests further investigation in several directions. First, in this paper we have let aside information asymmetry and heterogeneity in order to focus on the impact of strategic interactions on individual beliefs and from there on equilibrium prices and allocations. It would be useful to consider a more general model including both a strategic use of private information and a strategic choice of beliefs. It would be also useful to analyze how our results can be transposed in a dynamic setting. In particular, in the gurus framework it would be interesting to analyze in a dynamic setting how investors choose their guru and how they might decide to move from one guru to another one. Finally, in this paper we have only considered totally ordered families of possible subjective distributions for the risky asset payoffs. In particular, all beliefs deformations can be interpreted in terms of pessimism/optimism or in terms of doubt/overconfidence. It would be interesting to consider more general possible deformations of the objective distribution. For instance, in a complete markets framework, it should be interesting to take the set of all possible probability distributions as the strategic set of the agents and analyze the type of deformations induced by the strategic behavior in this setting.

Appendix Proofs

Proof of Proposition 1

1. For chosen beliefs $(\mu_i)_{i=1,2}$, the optimal demand for agent i is given by $\alpha_i (p, \mu_i) = \theta_i \frac{\mu_i - p}{\sigma^2}$, and the market clearing price is then given by $p(\mu_1, \mu_2) = \left(\frac{\theta_1}{\theta_1+\theta_2}\right)\mu_1 + \left(\frac{\theta_2}{\theta_1+\theta_2}\right)\mu_2 - \frac{\sigma^2}{\theta_1+\theta_2}$. The resulting expected utility from trade for agent i, given the belief μ_j of agent j, $j \neq i$, can

be written

$$
\begin{aligned}
U_i(\mu_i) &= E\left[-\exp-\frac{\frac{1}{2}p(\mu_1,\mu_2)+\alpha_i(\mu_1,\mu_2)(\widetilde{x}-p(\mu_1,\mu_2))}{\theta_i}\right] \\
&= -\exp\left[-\frac{\frac{1}{2}p(\mu_1,\mu_2)+\alpha_i(\mu_1,\mu_2)(\mu-p(\mu_1,\mu_2))}{\theta_i}+\frac{1}{2}\left(\frac{\alpha_i(\mu_1,\mu_2)}{\theta_i}\right)^2\sigma^2\right]
\end{aligned}
$$

where $\alpha_i(\mu_1,\mu_2)=\theta_i\frac{\mu_i-p(\mu_1,\mu_2)}{\sigma^2}$.

Maximizing this quantity amounts to maximizing

$$
A_i(\mu_i)=\frac{1}{2}p(\mu_1,\mu_2)+\alpha_1(\mu_1,\mu_2)(\mu-p(\mu_1,\mu_2))-\frac{1}{2}\frac{1}{\theta_1}(\alpha_1(\mu_1,\mu_2))^2\sigma^2.
$$

This program is concave and the maximum is reached for μ_i such that $\frac{dA_i}{d\mu_i}(\mu_i)=0$. This leads to

$$
\mu_i=\frac{2\mu\theta_j(\theta_1+\theta_2)+2\theta_i\theta_j\mu_j+\sigma^2(\theta_j-\theta_i)}{4\theta_j\theta_i+2\theta_i^2}. \tag{14}
$$

We solve then for $(\widehat{\mu}_1,\widehat{\mu}_2)$ and obtain Equations (4).

2. Straightforward using Equations (4).

3. Straightforward using 2. and the expression of the risk premium which, as seen in Equation (3), is given in the setting with heterogeneous beliefs by

$$
RP=\mu-p=\frac{\sigma^2}{\theta_1+\theta_2}+\left(\mu-\frac{\theta_1\widehat{\mu}_1+\theta_2\widehat{\mu}_2}{(\theta_1+\theta_2)}\right).
$$

4. The optimal demand of agent i is given by $\alpha_i(p(\mu_1,\mu_2),\mu_i)$, hence, using 3., we get

$$
\alpha_i^*=\frac{\theta_i}{\overline{\theta}}+\frac{\theta_i}{\sigma^2}\left(\widehat{\mu}_i-\frac{\theta_1\widehat{\mu}_1+\theta_2\widehat{\mu}_2}{\overline{\theta}}\right).
$$

Using Equation (6) , this leads to

$$
\alpha_i^*=\frac{\theta_i}{\overline{\theta}}+\frac{(\theta_j-\theta_i)}{4\overline{\theta}}.
$$

5. The utility level $U_1(\widehat{\mu}_1)$ given $\widehat{\mu}_2$ is given by

$$
\begin{aligned}
U_1\left(\widehat{\mu}_1\right) &= \theta_1 \frac{\widehat{\mu}_1 - p(\widehat{\mu}_1, \widehat{\mu}_2)}{\sigma^2}(\mu - p(\widehat{\mu}_1, \widehat{\mu}_2)) + \frac{1}{2} p(\widehat{\mu}_1, \widehat{\mu}_2) - \frac{1}{2}\frac{1}{\theta_1}\left[\theta_1 \frac{\widehat{\mu}_1 - p(\widehat{\mu}_1, \widehat{\mu}_2)}{\sigma^2}\right]^2 \sigma^2 \\
&= -\frac{1}{32}\frac{\left(3\sigma^2\theta_2^3 - 16\mu\theta_1^3\theta_2 - 2\sigma^2\theta_1^3 - 16\mu\theta_1\theta_2^3 + 8\sigma^2\theta_1\theta_2^2 + 7\sigma^2\theta_1^2\theta_2 - 32\mu\theta_1^2\theta_2^2\right)}{\overline{\theta}^2\theta_1\theta_2}
\end{aligned}
$$

while the Walrasian equilibrium utility level is given by

$$
\begin{aligned}
U_1\left(\mu\right) &= \theta_1 \frac{\mu - p(\mu, \mu)}{\sigma^2}\mu + \left(\frac{1}{2} - \theta_1 \frac{\mu - p(\mu, \mu)}{\sigma^2}\right) p(\mu, \mu) - \frac{1}{2}\frac{1}{\theta_1}\left[\theta_1 \frac{\mu - p(\mu, \mu)}{\sigma^2}\right]^2 \sigma^2 \\
&= \frac{1}{4\theta_1\theta_2 + 2\theta_1^2 + 2\theta_2^2}\left(2\mu\theta_1\theta_2 - \sigma^2\theta_2 + \mu\theta_1^2 + \mu\theta_2^2\right).
\end{aligned}
$$

We obtain $U_2\left(\widehat{\mu}_2\right)$ given $\widehat{\mu}_1$ as well as $U_2\left(\mu\right)$ similarly. The difference, for each agent, between the Nash equilibrium utility level and the Walrasian equilibrium utility level is given by

$$
U_i\left(\widehat{\mu}_i\right) - U_i\left(\mu\right) = \left(-\frac{1}{32}\right)\overline{\theta}^{-2}\theta_1^{-1}\theta_2^{-1}\left(\theta_j - \theta_i\right)^2\left(3\theta_j - 2\theta_i\right)\sigma^2.
$$

It is clear then that both quantities are negative for $\frac{2}{3}\theta_2 < \theta_1 < \frac{3}{2}\theta_2$. ■

Proof of Proposition 2

1. and 4. Taking the beliefs $(\widehat{\mu}_j)_{j\neq i}$ of the other agents as given, agent i maximizes

$$
A_i\left(\mu_i\right) = \frac{1}{2}p + \alpha_i\left(\mu - p\right) - \frac{1}{2}\frac{1}{\theta_i}\left(\alpha_i\right)^2\sigma^2
$$

where p and α_i both depend on μ_i and on $(\widehat{\mu}_j)_{j\neq i}$. They are given by $\alpha_i = \theta_i \frac{\mu_i - p}{\sigma^2}$, and $p = \frac{\theta_i}{\overline{\theta}}\mu_i + \sum_{j\neq i}\frac{\theta_j}{\overline{\theta}}\widehat{\mu_j} - \frac{\sigma^2}{\overline{\theta}}$.

Setting $\frac{dA_i}{d\mu_i}\left(\widehat{\mu}_i\right) = 0$ leads to

$$
\begin{aligned}
& N\widehat{\mu}_i\left(\sum_{j\neq i}\theta_j\right)\left(2\theta_i + \sum_{j\neq i}\theta_j\right) \\
&= \sigma^2\left[\sum_{j\neq i}\theta_j - (N-1)\,\theta_i\right] + N\mu\left(\sum_{j\neq i}\theta_j\right)^2 + N\mu\theta_i\sum_{j\neq i}\theta_j + N\theta_i\left(\sum_{j\neq i}\theta_j\widehat{\mu}_j\right).
\end{aligned}
$$

The previous equation can be written

$$
N\widehat{\mu}_i\overline{\theta}^2 = \sigma^2\overline{\theta} - N\sigma^2\theta_i + N\mu\overline{\theta}^2 - N\mu\overline{\theta}\theta_i + N\theta_i\left(\sum_{j=1}^{N}\theta_j\widehat{\mu}_j\right). \tag{15}
$$

Multiplying Equations (15) by θ_i and summing for $i = 1, ..., N$, we get

$$N\left(\sum_{i=1}^{N}\theta_i\widehat{\mu}_i\right)\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right) - N\mu\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right) = \sigma^2\overline{\theta}^2 - N\sigma^2\sum_{i=1}^{N}\theta_i^2,$$

hence

$$\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\widehat{\mu}_i = \mu - \frac{\left(N\sum_{i=1}^{N}\theta_i^2 - \overline{\theta}^2\right)\sigma^2}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right)}. \tag{16}$$

Since $N\sum_{i=1}^{N}\theta_i^2 - \overline{\theta}^2$ is positive, the average belief $\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\mu_i$ is pessimistic.

Now, replacing the weighted average $\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\widehat{\mu}_i$ in (15) by its expression (8), we get that there exists a unique Nash equilibrium with strategic beliefs given by

$$\widehat{\mu}_i = \mu + \frac{\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2 - (N-1)\theta_i\overline{\theta}}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right)}, \qquad \text{for } i = 1, ..., N.$$

5. Using (7) and 4 we have

$$RP = RP^{stdd} + \left(\mu - \sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\widehat{\mu}_i\right) = RP^{stdd} + \frac{\left(N\sum_{i=1}^{N}\theta_i^2 - \overline{\theta}^2\right)\sigma^2}{N\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right)}.$$

2. and 3. Summing, for $i = 1, ...N$, Equations (15), we get that

$$\frac{1}{N}\sum_{i=1}^{N}\widehat{\mu}_i - \mu = \frac{\left(N\sum_{i=1}^{N}\theta_i^2 - \overline{\theta}^2\right)\sigma^2}{N^2\overline{\theta}\left(\overline{\theta}^2 - \sum_{i=1}^{N}\theta_i^2\right)}.$$

Since, according to 4., the quantity $\sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\widehat{\mu}_i - \mu$ is negative, we get first that the unweighted average $\frac{1}{N}\sum_{i=1}^{N}\widehat{\mu}_i$ is pessimistic. Moreover, we get that $\frac{1}{N}\sum_{i=1}^{N}\widehat{\mu}_i > \sum_{i=1}^{N}\frac{\theta_i}{\overline{\theta}}\widehat{\mu}_i$, which means that there is a positive correlation between pessimism and risk tolerance. ■

Proof of Proposition 3

1. A simple computation of the optimal strategies for both gurus leads to

$$\begin{aligned}
\mu_1 &= \frac{2\mu\Theta_1\Theta_2\theta_1^*\theta_2^* + 2\sigma^2\Theta_1\Theta_2\theta_2^* - \sigma^2\Theta_1\theta_1^*\theta_2^* - \sigma^2\Theta_2\theta_1^*\theta_2^* + 2\mu\Theta_2^2\theta_1^*\theta_2^* + \sigma^2\Theta_1^2\theta_2^* + \sigma^2\Theta_2^2\theta_1^*}{2\theta_2^*\theta_1^*\Theta_2\left(\Theta_1 + \Theta_2\right)}, \\
\mu_2 &= \frac{2\mu\Theta_1\Theta_2\theta_1^*\theta_2^* + 2\sigma^2\Theta_1\Theta_2\theta_1^* - \sigma^2\Theta_1\theta_1^*\theta_2^* - \sigma^2\Theta_2\theta_1^*\theta_2^* + 2\mu\Theta_1^2\theta_1^*\theta_2^* + \sigma^2\Theta_1^2\theta_2^* + \sigma^2\Theta_2^2\theta_1^*}{2\theta_2^*\theta_1^*\Theta_1\left(\Theta_1 + \Theta_2\right)}.
\end{aligned}$$

2. Let us assume that $\theta_1^* > \theta_2^*$ and that $|\mu_1 - \mu| = |\mu - \mu_2|$. This leads to $\mu_1 = \mu_2$ or to

$\mu_1 - \mu = \mu - \mu_2$.

The first equation has no solution and the second one is equivalent to

$$F(\Theta_1) \equiv 2\Theta_1^3\theta_2^* - 2\Theta_1^3\theta_1^* - \Theta^3\theta_1^* + 3\Theta\Theta_1^2\theta_1^* - 3\Theta\Theta_1^2\theta_2^* + \Theta^2\theta_1^*\theta_2^* = 0.$$

It is easy to check that F is decreasing on $(0, \Theta)$ and that $F(0) < 0$ and $F(\Theta) > 0$ only if $\theta_1^* > \Theta > \theta_2^*$. This last condition means that one guru is more risk tolerant than the average while the other one is less risk tolerant than the average. Under this condition, the equation $F(\Theta_1) = 0$ admits only one solution in $(0, \Theta)$.

3. Let us assume now that $\Theta = \frac{\theta_1^*+\theta_2^*}{2}$ or in other words that the gurus are on average as risk tolerant as the agents in the economy. Under this assuption we obtain $F(\frac{\theta_1^*+\theta_2^*}{4}) < 0$ and $F(\frac{\theta_1^*+\theta_2^*}{2}) > 0$ which implies that $\Theta_1 \in \left(\frac{\theta_1^*+\theta_2^*}{4}, \frac{\theta_1^*+\theta_2^*}{2}\right)$ and $\Theta_2 \in \left(0, \frac{\theta_1^*+\theta_2^*}{4}\right)$. In particular, we have $\Theta_1 > \Theta_2$ which means that Group 1 is more risk tolerant than Group 2. Let us prove that Group 1is more pessimistic than Group 2 or equivalently that $\mu_1 - \mu < 0$. We have $\mu_1 - \mu = \frac{1}{2}\frac{\theta_1^*\theta_2^* - 4\Theta_1\theta_2^* - 4\Theta_1\theta_1^* + 4\Theta_1^2 + (\theta_1^*)^2}{(2\Theta_1 - \theta_1^* - \theta_2^*)(\theta_1^*+\theta_2^*)\theta_2^*\theta_1^*}(\theta_2^* - \theta_1^*)\sigma^2$ and has then the same sign as $H(\Theta_1) \equiv \theta_1^*\theta_2^* - 4\Theta_1\theta_2^* - 4\Theta_1\theta_1^* + 4\Theta_1^2 + (\theta_1^*)^2$ where Θ_1 solves $F(\Theta_1) = 0$. We check that $H\left(\frac{\theta_1^*+\theta_2^*}{2}\right) < 0$ and that $F(\underline{\Theta}) > 0$ where $\underline{\Theta}$ is the smallest solution of $H(\Theta) = 0$. This implies that $\Theta_1 \in \left(\underline{\Theta}, \frac{\theta_1^*+\theta_2^*}{2}\right)$ or that $H(\Theta_1) < 0$. Group 1 and guru 1 are then more pessimistic then Group 2 and guru 2. Pessimism and risk tolerance are then positively correlated at the gurus level.

4. If the two groups are of equal size, the average belief is then the objective belief. The average risk tolerance in Group 1 (resp. Group 2) is then equal to $2\Theta_1$ (resp. $2\Theta_2$) and the covariance between optimism and risk tolerance is then given by $(\Theta_1 - \Theta_2)(\mu_1 - \mu) < 0$. ■

Proof of Proposition 4

1. Agent i maximizes

$$A_i(\sigma_1, \sigma_2) = \frac{1}{2}p + \alpha_i(\mu - p) - \frac{1}{2}\frac{1}{\theta_i}(\alpha_i)^2\sigma^2$$

with respect to σ_i where p and α_i both depend on σ_i and are given by $p = \mu - \left(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2}\right)^{-1}$ and $\alpha_i = \theta_i\frac{\mu - p}{\sigma_i^2}$.

This problem is concave in $\left(\frac{\theta_1}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2}\right)^{-1}$. Setting $\frac{dA_i}{d\sigma_i}(\widehat{\sigma}_1, \widehat{\sigma}_2) = 0$ leads to

$$\widehat{\sigma}_i^2 = \frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_i}{\theta_j}\widehat{\sigma}_j^2\right). \tag{17}$$

The optimal beliefs are then given by Equations (9). Equation (10) follows.

Since

$$\left|\left(\frac{\theta_1-\theta_2}{4\theta_2}\right)\left(\frac{4\theta_1}{\theta_2-\theta_1}\right)\right|=\frac{\theta_1}{\theta_2}$$

the more risk tolerant agent exhibits more doubt than the less risk tolerant agent exhibits overfidence.

2. Straightforward using Equations (9).

3. Immediate in view of 2. ■

Proof of Proposition 5

It is well known that due to Inada conditions, the demand function is characterized by the following first order condition

$$E^{\mu}\left[(\tilde{x}-p)u_i'(\alpha_i(p,\mu)(\tilde{x}-p)+\frac{1}{2}p)\right]=0$$

and the partial derivatives of $\alpha_i(p,\mu)$ with respect to p and μ are given by

$$\begin{aligned}
\frac{\partial\alpha_i}{\partial p}(p,\mu) &= -\frac{E^{\mu}\left[\left(\frac{1}{2}-\alpha_i(p,\mu)\right)(\tilde{x}-p)u_i''(c(p,\mu))-u_i'(c(p,\mu))\right]}{E^{\mu}\left[(\tilde{x}-p)^2u_i'(c(p,\mu))\right]}\\
\frac{\partial\alpha_i}{\partial \mu}(p,\mu) &= -\frac{\frac{\partial}{\partial\mu}E^{\mu}\left[(\tilde{x}-p)u_i'(c(p,\mu))\right]\Big|_{(\mu,p,\alpha_i(p,\mu))}}{E^{\mu}\left[(\tilde{x}-p)^2u_i'(c(p,\mu))\right]}.
\end{aligned}$$

with $c(p,\mu)=\alpha_i(p,\mu)(\tilde{x}-p)+\frac{1}{2}p$. Letting $\widetilde{c}$ denote $c(p,\mu)$, remark that

$$E^{\mu}\left[\left(\frac{1}{2}-\alpha\right)(\tilde{x}-p)u_i''(\widetilde{c})-u_i'(\widetilde{c})\right]=E^{\mu}\left[-u_i'(\widetilde{c})-\tilde{y}u_i''(\widetilde{c})+\frac{1}{2}\widetilde{c}u_i''(\widetilde{c})\right].$$

Hence, $\frac{\partial\alpha_i}{\partial p}(p,\mu)$ is negative. Furthermore, $(\tilde{x}-p)u_i'(\widetilde{c})=\frac{1}{\alpha}\widetilde{c}u'(\widetilde{c})-\frac{1}{2}\frac{p}{\alpha}u'(\widetilde{c})$ and is then increasing. By the first-stochastic dominance property, we have $\frac{\partial\alpha_i}{\partial\mu}(p,\mu)\geq 0$.

We have then

$$\frac{\partial p}{\partial\mu_i}=-\frac{\frac{\partial\alpha_i}{\partial\mu_i}(p,\mu_i)}{\frac{\partial\alpha_1}{\partial p}(p,\mu_1)+\frac{\partial\alpha_2}{\partial p}(p,\mu_2)}$$

hence $\frac{\partial p}{\partial\mu_i}\geq 0$, $i=1,2$.

For $i\neq j$, we have $\frac{\partial\alpha_i^*(\mu_1,\mu_2)}{\partial\mu_i}=\frac{\partial\alpha_i(p,\mu_i)}{\partial\mu_i}+\frac{\partial\alpha_i(p,\mu_i)}{\partial p}\frac{\partial p}{\partial\mu_i}=\frac{\frac{\partial\alpha_i}{\partial\mu_i}(p,\mu_i)\frac{\partial\alpha_j}{\partial p}(p,\mu_j)}{\frac{\partial\alpha_1}{\partial p}(p,\mu_1)+\frac{\partial\alpha_2}{\partial p}(p,\mu_2)}\geq 0.$

If Problem (11) admits an interior solution, the first-order condition for agent i gives

$$E\left[(\frac{\partial\alpha_i^*}{\partial\mu_i}(X-p)+\left(\frac{1}{2}-\alpha_i^*\right)\frac{\partial p}{\partial\mu_i})u_i'(\alpha_i^*(X-p)+\frac{1}{2}p)\right]=0.$$

If $\frac{1}{2}-\alpha_i^*(\mu_1,\mu_2)\leq 0$ then $\left(\frac{1}{2}-\alpha_i^*(\mu_1,\mu_2)\right)\frac{\partial p}{\partial \mu_i}\leq 0$, hence $E\left[(X-p\left(\mu_1,\mu_2\right))u_i'\right]\geq 0$. As previously, by the first-stochastic dominance property we obtain $\mu_i^*\leq 0$. Analogously $\frac{1}{2}-\alpha_i^*(\mu_1,\mu_2)\geq 0$ leads to $\mu_i^*\geq 0$. We have then proved that the agent for which $\alpha_i^*(\mu_1,\mu_2)\geq\frac{1}{2}$ (resp. $\alpha_i^*(\mu_1,\mu_2)\leq\frac{1}{2}$) is pessimistic (resp. optimistic).

If one of the utility functions (let us say u_1) is more risk averse than the other one in the sense of Arrow-Pratt, let us prove that $\alpha_1^*(\mu_1,\mu_2)\leq\frac{1}{2}$. If this is not the case, we have $\alpha_2^*(\mu_1,\mu_2)\leq\frac{1}{2}$ and agent 2 is optimistic while agent 1 is pessimistic. We have then

$$\frac{1}{2}<\alpha_1(p\left(\mu_1,\mu_2\right),\mu_1)\leq\alpha_2(p\left(\mu_1,\mu_2\right),\mu_1)$$

because agent 1 is more risk averse. Furthermore we have $\alpha_2(p\left(\mu_1,\mu_2\right),\mu_1)\leq\alpha_2(p\left(\mu_1,\mu_2\right),\mu_2)$ because μ_2 is larger than μ_1. We would have then $\alpha_2^*(\mu_1,\mu_2)>\frac{1}{2}$ which contradicts our assumption. ■

Proof of Proposition 6

1. As in the previous proofs, direct computations lead to the following equilibrium prices and quantities in a Walrasian setting when agents are endowed with beliefs (μ_i,ν_i):

$$\left\{\begin{array}{c} p=\sum_{i=1}^2\frac{\theta_i}{\overline{\theta}}\mu_i-\frac{\sigma^2+\sigma\varpi\rho}{\overline{\theta}},\qquad q=\sum_{i=1}^2\frac{\theta_i}{\overline{\theta}}\nu_i-\frac{\varpi^2+\sigma\varpi\rho}{\overline{\theta}}\\ \alpha_1=\theta_1\frac{\mu_1-p}{\sigma^2(1-\rho^2)}-\theta_1\frac{(\nu_1-q)\rho}{\sigma\varpi(1-\rho^2)}=\frac{\theta_1}{\overline{\theta}}\left[1+\frac{\theta_2(\mu_1-\mu_2)}{\sigma^2(1-\rho^2)}-\frac{\theta_2(\nu_1-\nu_2)}{\sigma\varpi(1-\rho^2)}\right]\\ \beta_1=\theta_1\frac{\nu_1-q}{\varpi^2(1-\rho^2)}-\theta_1\frac{(\mu_1-p)\rho}{\sigma\varpi(1-\rho^2)}=\frac{\theta_1}{\overline{\theta}}\left[1+\frac{\theta_2(\nu_1-\nu_2)}{\varpi^2(1-\rho^2)}-\frac{\theta_2(\mu_1-\mu_2)}{\sigma\varpi(1-\rho^2)}\right]\end{array}\right. . \tag{18}$$

In the setting of the proposition, agent i maximizes

$$A_i\left(\mu_1,\nu_1,\mu_2,\nu_2\right)=\frac{1}{2}p+\alpha_i\left(\mu-p\right)+\frac{1}{2}q+\beta_i\left(\nu-q\right)-\frac{1}{2}\frac{1}{\theta_i}\left(\alpha_i^2\sigma^2+\beta_i^2\varpi^2+2\alpha_i\beta_i\sigma\varpi\rho\right).$$

with respect to (μ_i,ν_i) taking $\left(\mu_j,\nu_j\right)=\left(\widehat{\mu}_j,\widehat{\nu}_j\right)$, $j\neq i$, as given. The maximization programs under consideration are concave. Setting $\frac{dA_i}{d\mu_i}=\frac{dA_i}{d\nu_i}=0$ leads to

$$\widehat{\mu}_i=\mu-\frac{\left(\theta_i-\theta_j\right)\left(\sigma^2+\sigma\varpi\rho\right)}{4\theta_j\overline{\theta}},\qquad \widehat{\nu}_i=\nu-\frac{\left(\theta_i-\theta_j\right)\left(\varpi^2+\sigma\varpi\rho\right)}{4\theta_j\overline{\theta}}$$

which is the unique solution of the Nash equilibrium in beliefs. At the aggregate level (i.e. $\widehat{\mu}_i+\widehat{\nu}_i$), we check that the more risk tolerant agent is pessimistic and the less risk tolerant is optimistic. Besides, the more risk tolerant agent is more pessimistic than the less risk tolerant agent is optimistic. These properties are inherited at the individual assets level as far as σ^2+

$\sigma\varpi\rho \geq 0$ and $\varpi^2 + \sigma\varpi\rho \geq 0$.

2. and 3. Straightforward using the result of 1. as well as Equations (18). ■

Proof of Proposition 7

The utility level of agent i is given by $E_i\left[u_i\left(\frac{1}{2}p + \alpha_i(p,\mu_i)\left(\widetilde{x} - p\right)\right)\right]$ with $\alpha_i(p,\mu_i) = \theta_i\frac{\mu_i - p}{\sigma^2}$. Then, for a given p, the agent maximizes $\theta_i\frac{(\mu_i - p)^2}{\sigma^2}$.

When $p > \frac{a+b}{2}$, all the agents have the same belief a and the equilibrium price, if it exists, must satisfy $\overline{p} = a - \frac{\sigma^2}{\theta_1+\theta_2}$ which is not compatible with the condition $p > \frac{a+b}{2}$. When $p < \frac{a+b}{2}$, all the agents have the same belief b and the equilibrium price, if it exists, must satisfy $\overline{p} = b - \frac{\sigma^2}{\theta_1+\theta_2}$ which is compatible with the condition $p < \frac{a+b}{2}$ only if $\frac{\sigma^2}{\theta_1+\theta_2} > \frac{b-a}{2}$. In this equilibrium all agents are optimistic and the risk premium is lower than in the standard rational expectations equilibrium. When $p = \frac{a+b}{2}$, both agents may choose the same belief b leading to an equilibrium only if $\frac{\sigma^2}{\theta_1+\theta_2} = \frac{b-a}{2}$. They may also choose different beliefs. If agent 1 (resp. 2) chooses a (resp. b), the market clearing condition leads to

$$\frac{\theta_1 a + \theta_2 b}{\theta_1 + \theta_2} - \frac{\sigma^2}{\theta_1 + \theta_2} = \frac{a+b}{2} \tag{19}$$

which implies that the more risk tolerant agent is the more optimistic one and furthermore imposes a relationship between $\sigma^2, \theta_1, \theta_2, a$ and b. If $\mu = \frac{a+b}{2}$ and $\theta_1 < \theta_2$, then $\frac{\theta_1 a+\theta_2 b}{\theta_1+\theta_2} > \mu$. This leads to a lower risk-premium. ■

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